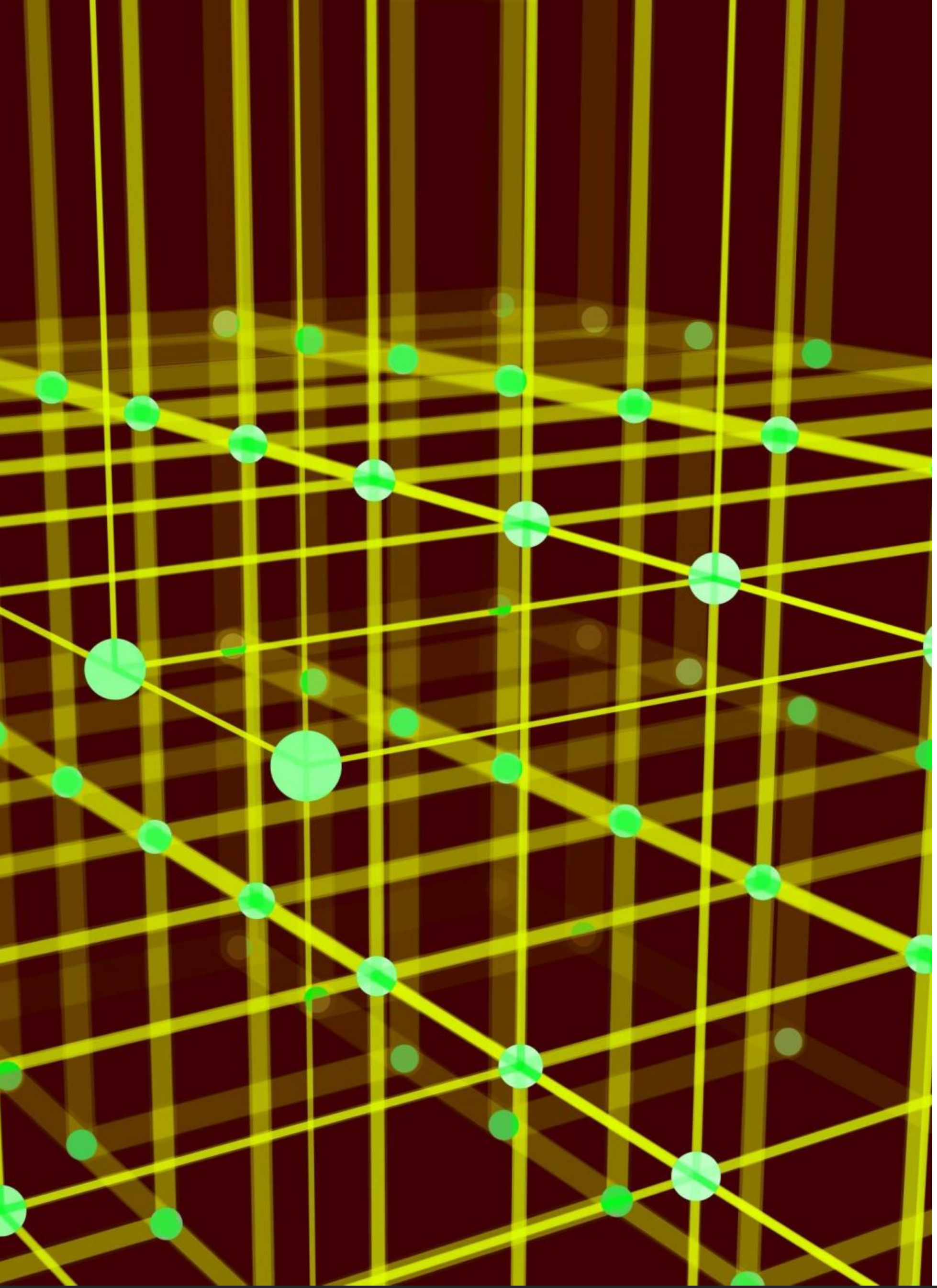

Spatial Random Processes

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1.13 (p. 444) Let X be an $N(0, 1)$-distributed r.v., σ, b real numbers and $x, K > 0$. Show that

$$\mathrm{E}[(xe^{b+\sigma X} - K)^+] = xe^{b+\frac{1}{2}\sigma^2}\Phi(-\zeta + \sigma) - K\Phi(-\zeta) ,$$

where $\zeta = \frac{1}{\sigma}(\log \frac{K}{x} - b)$ and Φ denotes the partition function of an $N(0, 1)$-distributed r.v. This quantity appears naturally in many questions in mathematical finance, see Sect. 13.6. (x^+ denotes the positive part function, $x^+ = x$ if $x \geq 0$, $x^+ = 0$ if $x < 0$.)

X isn't just any random variable, it follows a normal distribution, also known as a bell curve. This means there's a higher chance of X being close to the average (often zero).

Question 1

Proof 1

Step 1: Initial Expression

The initial expression for the expected value is given by:

$$E[(xe^{b+\sigma z} - K)^+] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (xe^{b+\sigma z} - K)^+ e^{-z^2/2}\, dz$$

The term $(xe^{b+\sigma z} - K)^+$ denotes the positive part of $xe^{b+\sigma z} - K$, which is zero if $xe^{b+\sigma z} - K \leq 0$.

The integral sign with infinity as the upper bound means we are integrating the entire function from negative infinity to positive infinity. This indicates that we are considering all possible values of z.

Proof 1

Step 2: Defining ζ

The function inside the integral is zero when $xe^{b+\sigma z} - K \leq 0$. Solving for z, we get:

$$z \leq \zeta := \frac{1}{\sigma}\left(\log\frac{K}{x} - b\right)$$

Thus, the integral can be split into two parts: one from $-\infty$ to ζ where the integrand is zero, and one from ζ to $+\infty$.

We are defining the condition under which the function inside the integral equals zero. This condition is then used to split the integral into two parts, which can then be evaluated separately.

Proof 1

Step 3: Changing the Integration Limits

Since the integral is zero for $z \leq \zeta$, the integral reduces to:

$$E[(xe^{b+\sigma z} - K)^+] = \frac{1}{\sqrt{2\pi}} \int_{\zeta}^{+\infty} (xe^{b+\sigma z} - K)e^{-z^2/2}\, dz$$

Step 4: Splitting the Integral

We split the integral into two parts:

$$= \frac{x}{\sqrt{2\pi}} \int_{\zeta}^{+\infty} e^{b+\sigma z - z^2/2}\, dz - \frac{K}{\sqrt{2\pi}} \int_{\zeta}^{+\infty} e^{-z^2/2}\, dz$$

Proof 1

Step 5: Simplifying the First Integral

For the first integral, we complete the square in the exponent $b + \sigma z - \frac{z^2}{2}$:

$$b + \sigma z - \frac{z^2}{2} = -\frac{1}{2}(z - \sigma)^2 + \frac{1}{2}\sigma^2 + b$$

Thus,

$$e^{b+\sigma z - z^2/2} = e^{b+\sigma^2/2} \cdot e^{-(z-\sigma)^2/2}$$

Then the integral becomes:

$$= \frac{xe^{b+\sigma^2/2}}{\sqrt{2\pi}} \int_{\zeta}^{+\infty} e^{-(z-\sigma)^2/2}\, dz$$

Step 6: Recognizing the CDF of Normal Distribution

The integral $\int_{\zeta}^{+\infty} e^{-(z-\sigma)^2/2}\, dz$ is related to the cumulative distribution function (CDF) of the standard normal distribution, $\Phi(z)$. Shifting the limits:

$$\int_{\zeta}^{+\infty} e^{-(z-\sigma)^2/2}\, dz = \int_{\zeta-\sigma}^{+\infty} e^{-u^2/2}\, du = \sqrt{2\pi}\Phi(\sigma - \zeta)$$

Thus, the first term becomes:

$$\frac{xe^{b+\sigma^2/2}}{\sqrt{2\pi}} \int_{\zeta-\sigma}^{+\infty} e^{-u^2/2}\, du = xe^{b+\sigma^2/2}\Phi(\sigma - \zeta)$$

Proof 1

Step 7: Simplifying the Second Integral

The second integral is directly the CDF of the normal distribution evaluated at ζ:

$$\frac{K}{\sqrt{2\pi}} \int_{\zeta}^{+\infty} e^{-z^2/2}\, dz = K\Phi(-\zeta)$$

Step 8: Final Expression

Combining both parts, we get the final expression:

$$E[(xe^{b+\sigma z} - K)^+] = xe^{b+\frac{1}{2}\sigma^2}\Phi(\sigma - \zeta) - K\Phi(-\zeta)$$

Question 2

Let B be a Brownian motion and let $s \leq t$.

Compute $\mathrm{E}[B_s B_t^2]$.

The question wants to find the average of the product of the values of B at two different times s and t, where s happens before t.

Proof 2

1. **Recall the properties of Brownian motion:**
   - B_t is normally distributed with mean 0 and variance t.
   - For $s \leq t$, $B_t = B_s + (B_t - B_s)$ where $B_t - B_s$ is independent of B_s and normally distributed with mean 0 and variance $t - s$.

2. **Express B_t^2 in terms of B_s and $B_t - B_s$:**

$$B_t^2 = (B_s + (B_t - B_s))^2 = B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2$$

3. **Take the expectation:**

$$E[B_s B_t^2] = E[B_s(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2)]$$

4. **Separate the expectation using linearity:**

$$E[B_s B_t^2] = E[B_s^3] + 2E[B_s^2(B_t - B_s)] + E[B_s(B_t - B_s)^2]$$

Proof 2

5. **Evaluate each term separately:**

- $E[B_s^3] = 0$ because B_s is normally distributed with mean 0.
- $E[B_s^2(B_t - B_s)] = E[B_s^2]E[B_t - B_s] = 0$ because $B_t - B_s$ is independent of B_s and has mean 0.
- $E[B_s(B_t - B_s)^2]$:

$$E[B_s(B_t - B_s)^2] = E[B_s]E[(B_t - B_s)^2] = (t - s)E[B_s] = 0$$

because $E[B_s] = 0$.

Since the Brownian motion at time s and Brownian motion at time t are statistically the same, then their expected value, E[B_sB_t^2] would also be the same.

Question 3

Let B be a Brownian motion. Compute

$$\lim_{t\to+\infty} \mathrm{E}[1_{\{B_t \le a\}}].$$

Proof 3

We have

$$
\begin{aligned}
E[1_{\{B_t \le a\}}] &= P(B_t \le a) \\
&= P(\sqrt{t}B_1 \le a) \\
&= P\left(B_1 \le \frac{a}{\sqrt{t}}\right)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\lim_{t \mapsto +\infty} E[1_{\{B_t \le a\}}] &= P\left(B_1 \le 0\right) \\
&= \frac{1}{2}.
\end{aligned}
$$

Since Brownian motion has no preference for positive or negative infinity, and it will eventually reach all positions with enough time, the probability of finding the particle at a specific value (a) or below (less than or equal to a) approaches 1/2 as time goes to positive infinity.

Question 4

4.10 (p. 479) Let $(E, \mathscr{E})$, $(G, \mathscr{G})$ be measurable spaces, X an E-valued r.v. such that

$$X = \phi(Y) + Z\,,$$

where Y and Z are independent r.v.'s with values respectively in E and G and where $\phi : G \to E$ is measurable.

Show that the conditional law of X given $Y = y$ is the law of $Z + \phi(y)$.

Proof 4

Theorem 1 *Let us denote by v_y, u_z, respectively, the laws of Y and Z and by μ_y the law of $(y)+Z$. We must prove that for every pair of bounded measurable functions*

$f : R \to R$ *and* $g : G \to R$ *we have*

$$E[f(X)g(Y)] = \int (v_y(v)f(x)d\mu_y(x))dv(y) \tag{1}$$

But we have

$$[f(x)d\mu_y(x) = \int f((y) + z)du_z(z) \tag{2}$$

and

$$E[f(X)g(Y)] = E[f((Y)+Z)g(Y)] = \int g(y)dv_y(y) \int f((y)+z)du_z(z) = \int f(x)dv_y(v)f(x)du_y(x). \tag{3}$$


Thank you